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STABILITY OF THE PINCH

by

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I. INTRODUCTION

In this report, the stability of a pinched fluid is studied theoretically, under the combined influence of a longitudinal magnetic field and conducting exterior shell. Conditions allowing complete stability are found.

The stabilizing effect of a conducting shell seems first to have been pointed out in 1953 (Tuck, Wash-146), and calculations of the general stabilizing effect of a longitudinal field in 1953 (Kruskal and Tuck, LA-1716), whose notation we use in this report. Low power experiments (Wash-184, 1954) gave no indication of beneficial effects from a longitudinal field, though it was realized that the pinch currents then available were scarcely adequate.

Some unclassified qualitative calculations of the conducting shell were also available in 1954 (Bostick, Levine, et al., Tufts College, Note #14). An expression using magnetohydrodynamics for the combined effect of longitudinal field and conducting shell was obtained in 1954 (Kruskal, PMS-16) and independently, Rosenbluth, UCRL-1954; but its implications not fully realized at the time. More recently (December, 1954), special advantages of an internal trapped longitudinal field and a hollow cylindrical plasma were pointed out by Levine and Combs, Tufts College, Note #16, although their quantitative results disagree with those derived in this paper. At the Los Alamos Christmas, 1955, meeting, discussions between

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Colgate, Kruskal, Rosenbluth, Teller, and Tuck also emphasized this point. At the insistence of Colgate, 1956, the expression described above was evaluated, and this report presents these results. We also will calculate the stability criterion without the magnetohydrodynamics approximation.

The geometry considered (see Fig. 1) is that of an infinite cylinder along the z-axis. Inside radius r_o is the confined plasma at uniform pressure and density and containing uniform longitudinal field of magnitude $\alpha_p B_o$. Outside the plasma is a region of vacuum extending to the perfect external conductor at radius βr_o. In the vacuum is a field with azimuthal component $B_o \frac{r_o}{r}$ and longitudinal component $\alpha_V B_o$. We consider perturbations which vary like $\left[e^{i(kz \pm m\varphi)}\right]$. Our object is to find stable configurations, i.e., to discover for what values of the parameters α_p, α_V and β the pinch is stable for all m and k.

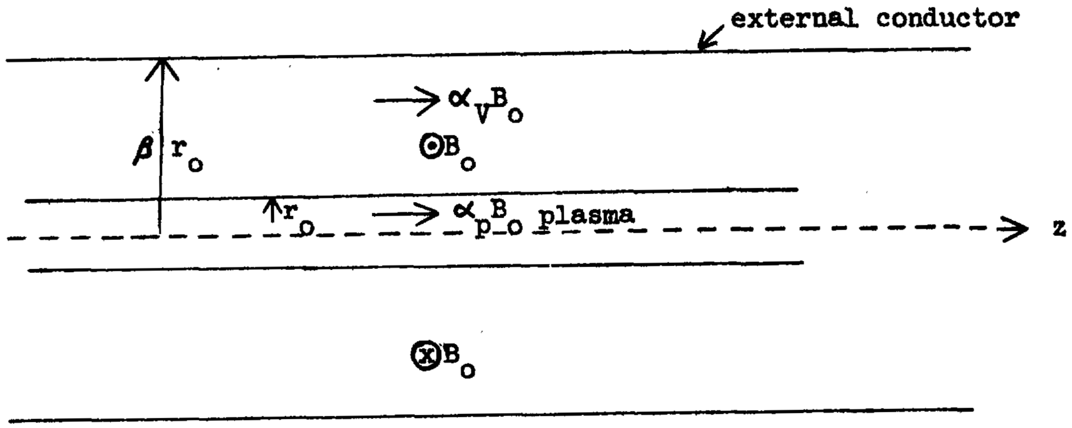

Fig. 1 Geometry considered in this report.

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The method used to calculate the stability is to make the relevant virtual displacement and note whether the resulting pressure distribution at the plasma surface is such as to cause the displacement to grow. This is equivalent to calculating the change of energy of the system. Our resulting expression in the magnetohydrodynamic case is identical with that of Kruskal. It says that for a given set of parameters, α_p, α_V, β, m, $Y = kr_o$, the pinch is stable if:

$$\alpha_p^2 Y^2 K_m(Y) + (m \pm \alpha_V Y)^2 \frac{G_{\beta,m}(Y)\, K_m(Y) - L_m(Y)}{1 - G_{\beta,m}(Y)} > 1 \tag{1}$$

K_m, L_m, and $G_{\beta,m}$ are defined in terms of Bessel functions as

$$K_m(Y) = \frac{J_m(iY)}{iY\, J_m'(iY)}, \qquad L_m(Y) = \frac{H_m(iY)}{iY\, H_m'(iY)}$$

$$G_{\beta,m}(Y) = \frac{H_m'(i\beta Y)}{H_m'(iY)} \quad \frac{J_m'(iY)}{J_m'(i\beta Y)} \tag{1'}$$

The particle picture leads to a slightly different form

$$\left(1 + \frac{4\pi(p_1 - p_3)}{\alpha_p^2\, B_o^2}\right) \alpha_p^2 Y^2 K_m(\gamma Y) + (m \pm \alpha_V Y)^2 \frac{G_{\beta,m}(Y)\, K_m(Y) - L_m(Y)}{1 - G_{\beta,m}(Y)} > 1 \tag{2}$$

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Here p_1 and p_3 are components of the pressure tensor along and perpendicular to the field respectively and

$$\gamma = \sqrt{\frac{(\alpha_p^2 B_o^2/4\pi) + p_1 - p_3}{(\alpha_p^2 B_o^2/4\pi) + Sp_3}}$$

$$S = \frac{1}{2} \frac{\int_o^1 \partial f/\partial\mu \left[(1 - \mu^2)^2/\mu\right] d\mu}{\int_o^1 f(\mu)\,(1 - \mu^2)\,d\mu} \qquad (3)$$

where $f(\mu)$ represents the initial angular distribution of particle velocity relative to the z-axis. For an initially isotropic distribution $S = p_1 - p_3 = 0$ and Eq. 2 becomes identical with Eq. 1. In practice, it is likely that $p_3 > p_1$ and Eq. 2 is then somewhat less favorable than Eq. 1. This is discussed in more detail at the end of Section V.

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II. DERIVATION OF THE EQUATION OF MOTION

The equation governing the field change is

$$\frac{1}{C}\frac{\partial \vec{B}}{\partial t} = -\nabla \times E = +\nabla \times \left(\frac{\vec{v}}{C} \times \vec{B}\right) \tag{4}$$

The second equation in the plasma is the equation expressing the fact that particles are tied to field lines and move with the E x B drift. We define

$$\vec{\xi} = \int \vec{v}\, dt \tag{5}$$

Then we may integrate Eq. 4 to obtain for the change in field

$$\delta \vec{B} = \nabla \times (\vec{\xi} \times \vec{B}_1) \tag{6}$$

where B_1 is the unperturbed field. We are, of course, interested only in terms linear in the displacement, $\vec{\xi}$.

As we have mentioned earlier, our method is to make a virtual displacement which, within each medium, preserves the equilibrium condition. Thus in the vacuum

$$\vec{j} = \frac{\nabla \times \delta \vec{B}}{4\pi} = 0 \tag{7}$$

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and in the plasma

$$\left(\frac{\nabla \times \delta \vec{B}}{4\pi}\right) \times \vec{B}_i - \nabla \cdot \underline{p} = 0 \tag{8}$$

where $\underline{p}$ is the pressure tensor of the plasma.

In the magnetohydrodynamic approximation, Eq. 8 becomes

$$\left(\frac{\nabla \times \delta \vec{B}}{4\pi}\right) \times \vec{B}_i - \nabla p = 0. \tag{9}$$

When we consider non-scalar effects Eq. 8 becomes [1]

$$\left(\frac{\nabla \times \delta \vec{B}}{4\pi}\right) \times \vec{B}_i - \nabla p_3 + (p_1 - p_3)\frac{\partial \hat{e}_1}{\partial x_1} + \hat{e}_1 \left\{ \frac{\partial (p_1 - p_3)}{\partial x_1} - \frac{p_1 - p_3}{b}\frac{\partial b}{\partial x_1} \right\} \tag{10}$$

Here $\hat{e}_1$ is a unit vector along the field lines and b the magnitude of the field. The pressure tensor is of the well-known form

$$p = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_3 & 0 \\ 0 & 0 & p_3 \end{pmatrix} \tag{11}$$

1. See Los Alamos notes on ionized gases.

III. DETERMINATION OF THE PERTURBED VACUUM FIELD

From Eq. 7, we may write

$$
\begin{aligned}
\delta \vec{B} &= \nabla \psi \\
\nabla \cdot \delta \vec{B} &= \nabla^2 \psi = 0
\end{aligned}
\tag{12}
$$

The solution with the desired z and φ dependence is

$$
\psi = \left[A\, J_m\, (ikr) + B\, H_m\, (ikr) \right] e^{i \left[kz \pm m\varphi \right]} \tag{13}
$$

A and B are to be determined from the boundary condition at the external conductor and displaced plasma interface

$$
\delta \vec{B} \cdot \vec{n} = 0 \tag{14}
$$

or

$$
\begin{aligned}
&\delta B_r(\beta r_o) = 0 \\
&\delta B_r(r_o) + B_o \delta \left[- ik \alpha_V\, r_o \mp im \right] = 0
\end{aligned}
\tag{14'}
$$

where we have assumed the displaced plasma interface is given by

$$r = r_o \left[1 + \delta e^{i(kz \pm m\varphi)} \right]. \tag{15}$$

We may now use Eq. 14' to determine the coefficients A and B in Eq. 13.

$$B = - A \frac{J_m'(ik\beta r_o)}{H_m'(ik\beta r_o)}$$

$$A = \frac{B_o \, \delta/k \left[k\alpha_V r_o \pm m \right]}{J_m'(ikr_o) - \left[J_m'(ik\beta r_o)/H_m'(ik\beta r_o) \right] H_m'(ikr_o)} \tag{16}$$

Finally, we calculate the change in pressure on the plasma surface resulting from the displacement

$$\delta P_o = \delta\left(\frac{B^2}{8\pi}\right) = \frac{B_o}{4\pi}\left[\delta B_\phi + \alpha_V \delta B_z \right] - \frac{B_o^2 \delta}{4\pi} e^{i(kz \pm m\varphi)} \tag{17}$$

The last term in Eq. 17 results from the evaluation of $B_o \frac{r_o}{r}$ at the displaced radius.

Using Eqs. 1', 12, 13, and 16, we finally get

$$\delta P_o = \frac{B_o^2}{4\pi} e^{i(kz \pm m\varphi)} \left\{ - 1 + (m \pm \alpha_V Y)^2 \frac{G_{\beta,m}(Y) \, K_m(Y) - L_m(Y)}{1 - G_{\beta,m}(Y)} \right\} \tag{18}$$

We note that - 1 in Eq. 18 tends to give δP_o the opposite sign for the displacement. In other words, the pressure is smaller where

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the surface is bulged out, thus lending to instability. We must still, of course, calculate δP_i, the resultant pressure inside, in order to obtain the net restoring force.

IV. PLASMA EQUATIONS IN MAGNETOHYDRODYNAMIC APPROXIMATION

The equation of equilibrium in the plasma is given by Eqs. 6 and 9 as

$$\left\{ \nabla \times \nabla \times (\vec{\xi} \times \vec{B}_i) \right\} \times \vec{B}_i - \nabla \delta p = 0 \tag{19}$$

To evaluate δp we note

$$\begin{aligned} \frac{d\rho}{dt} &= -\rho_i \, \nabla \cdot \vec{v} \\ \frac{d}{dt}\left(p\rho^{-\gamma}\right) &= 0 \end{aligned} \tag{20}$$

so that we may find

$$\delta p = -\gamma p_i \, \nabla \cdot \xi \tag{21}$$

If we multiply Eq. 19 by $\vec{B}_i$, we obtain

$$\vec{B}_i \cdot \nabla \, \nabla \cdot \xi = 0$$

Since $\vec{B}_i$ in the plasma is in the z direction and $\nabla \cdot \xi$ for our perturbation varies as e^{kz}, we must have

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$$\delta p = \nabla \cdot \vec{\xi} = 0 \tag{22}$$

In considering the components of Eq. 19 normal to $\vec{B}$, it is useful to use the vector identity

$$\nabla \times (\vec{\xi} \times \vec{B}_i) = + (B_i \cdot \nabla)\xi - (\xi \cdot \nabla)\, B_i + \vec{B}_i \nabla \cdot \vec{\xi} - \vec{\xi}\, \nabla \cdot \vec{B}_i$$

In our case, this simplifies to

$$\delta \vec{B} = \nabla \times (\vec{\xi} \times \vec{B}_i) = + (B_i \cdot \nabla)\, \vec{\xi} \tag{23}$$

and Eq. 19 becomes

$$(B_i \cdot \nabla)\, (\nabla \times \vec{\xi}) = 0$$

since $B \cdot \nabla$ commutes with $\nabla \times$ and the other terms vanish.

As before, this implies

$$\nabla \times \vec{\xi} = 0 \tag{24}$$

Finally, Eqs. 22 and 24 imply

$$\vec{\xi} = C\, \nabla \left[J_m(ikr) e^{i(kz \pm m\varphi)} \right]$$

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The boundary condition on ξ_r is given by Eq. 15 so that

$$\vec{\xi} = \frac{r_o \delta}{ik\, J'_m(ikr)} \nabla \left[J_m(ikr)\, e^{i(kz \pm m\varphi)} \right] \tag{25}$$

From Eqs. 23 and 25

$$\delta B_z = B_i \left[\frac{-\, k r_o \delta}{i\, J'_m(ikr)} \right] J_m(ikr)\, e^{i(kz \pm m\varphi)}$$

Finally, we compute

$$\delta P_i = \frac{B_i}{4\pi} \delta B_z \text{ at } r = r_o$$

and recall that $B_i = \alpha_p B_o$ to obtain

$$\delta P_i = - \frac{\alpha_p^2 B_o^2 \delta}{4\pi} Y^2 K_m(Y)\, e^{i(kz \pm m\varphi)} \tag{26}$$

We are now prepared to write the stability condition

$\frac{\delta P_o - \delta P_i}{r - r_o} > 0$, and by using Eq. 18, we obtain Eq. 1

$$\alpha_p^2 Y^2 K_m(Y) + (m \pm \alpha_V Y)^2 \frac{G_{\beta,m}(Y)\, K_m(Y) - L_m(Y)}{1 - G_{\beta,m}(Y)} > 1$$

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V. ADIABATIC INVARIANT CALCULATION

In applying Eq. 10, the principal problem is the evaluation of the changes in the pressure tensor brought about by the deformation. These will be calculated by considering the effect of the displacement on the orbit of a single particle and then summing over all orbits.

Before the deformation, the number of particles per cubic centimeter per unit energy and solid angle is given by $g(E_o)f(\mu)dE_o d\mu$.

Here E_o and μ are the energy of the particles and the direction cosine of its velocity vector with the z-axis before the displacement.

The adiabatic invariants of the motion are the magnetic moment and the action integral (see footnote 1).

$$\frac{E_\perp}{B_i + \delta b(\ell)} = \text{Const} = \frac{E_o(1 - \mu^2)}{B_i} \tag{27}$$

$$\int \sqrt{E_{\parallel}}\, d\ell = \int \sqrt{E_o + \delta E - \left[B_i + \delta b(\ell)\right]\left[E_o(1 - \mu^2)/B_i\right]}\, d\ell = |\mu|\sqrt{E_o}\int d\ell \tag{28}$$

Here δE_o and $\delta b(\ell)$ are the changes induced in E and B by the displacement. $E_\perp$ and $E_\parallel$ are the orbital energies perpendicular and parallel to the field lines. $d\ell$ is the element of length along the line.

Relations 27 and 28 are related to conservation of angular momentum about the field direction, and conservation of linear

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momentum along the field lines. The integral on the left of Eq. 28 is to be extended over a complete period of the perturbation, unless $E_{\parallel}$ goes to zero in which case the integral is to be extended between the turning points of the orbit.

Now the variation in field strength is given by

$$\delta b = \mathcal{E}(r)\, B_i \cos\,(kz \pm m\varphi).$$

Along a given field, line r and φ are constant while z varies. Hence, in Eq. 28, we may use $\ell = (kz \pm m\varphi)$ and

$$\delta b(\ell) = \mathcal{E}\, B_i \cos \ell$$

so that Eq. 28 becomes

$$\int_{0 \text{ or } \cos^{-1}\left[\frac{1 + (\delta E/\mu^2 E_o}{(1 - \mu^2/\mu^2)\mathcal{E}}\right]}^{2\pi \text{ or } 2\pi - \cos^{-1}\left[\frac{1 + (\delta E/\mu^2 E_o)}{(1 - \mu^2/\mu^2)\mathcal{E}}\right]} \sqrt{\left[1 + (\delta E/\mu^2 E_o)\right] - \left[(1 - \mu^2)/\mu^2\right] \mathcal{E} \cos \ell}\; d\ell = 2\pi \qquad (29)$$

Eq. 29 is the determining equation for δE. The alternative limits on the integral are to be used if the $\sqrt{}$ would be negative at 0 or 2π.

Next we must write down the expression for the pressure as a sum over particles. The equilibrium equation in the direction $\hat{e}_1$, along the lines, is automatically satisfied by this approach, since it only means that the particles are moving in orbits consistent with the magnetic field. It is relatively trivial to verify this by direct computation, so we need only write down the expression for p_3. By definition, p_3 is the energy density per cubic centimeter in the plane $\perp$ to the field. The contribution of the particles in the range $dE_o d\mu_o$ will be just the density of such particles times their perpendicular energy, given by Eq. 27. The density of such particles is

$$\rho(E_o,\mu) = g(E_o)\ f(\mu)\ (1 + \varepsilon \cos \ell)\ \frac{(1/v_{\parallel})\int d\ell}{\int (d\ell/v_{\parallel})} \tag{30}$$

where

$$v_{\parallel} = \sqrt{1 + \frac{\delta E}{\mu^2 E_o} - \frac{(1-\mu^2)}{\mu^2}\,\varepsilon \cos \ell}$$

and the limits on the integral are the same as those in Eq. 29.

The factor $(1 + \varepsilon \cos \ell)$, the ratio of magnetic field to initial field, arises from the fact that particles are attached to field lines so that their density is proportional to the density of field lines, i.e., B. The last term in Eq. 30 is the fraction of its time which the particle spends at a given position of its orbit. Combining Eqs. 27 and 30, we get

$$p_3 = 2 \int_0^\infty E_o \, g(E_o) \, dE_o \int_{\bar{\mu}}^{1} d\mu f(\mu) \, (1 - \mu^2) \, (1 + \varepsilon \cos \ell)^2 \, \frac{2\pi}{v_\parallel \int d\ell / v_\parallel} \tag{31}$$

In writing Eq. 31, we have assumed $f(\mu)$ to be even. $\bar{\mu}$ is the angle at which $v_\parallel$ vanishes.

Next we introduce

$$\tau = \frac{1 + (\delta E)/(\mu^2 E_o)}{\left[(1 - \mu^2)/(\mu^2)\right]\varepsilon} \qquad \text{and obtain}$$

$$p_3 = 4\pi \int_0^\infty E_o \, g(E_o) \, dE_o \int_{\cos \ell}^{\infty} d\tau f(\mu) \, (1 - \mu^2) \, (1 + \varepsilon \cos \ell)^2$$

$$\frac{1}{\sqrt{1 - \cos \ell / \tau}} \, \frac{d\mu}{d\tau} \, \frac{1}{\int_{0 \text{ or } \cos^{-1}\tau} d\ell' / \sqrt{1 - (\cos \ell' / \tau)}} \tag{32}$$

and Eq. 29 becomes

$$I(\tau) = \frac{1}{2\pi} \int_{0,\cos^{-1}\tau}^{2\pi,\, 2\pi - \cos^{-1}\tau} \sqrt{\tau - \cos \ell'} \;\; d\ell' = \frac{1}{\sqrt{\left[(1 - \mu^2)/(\mu^2)\right]\varepsilon}}. \tag{33}$$

From Eq. 33, we get

$$\frac{d\mu}{d\tau} = \frac{1}{4\pi}\left(1 - \mu^2\right)^{3/2} \frac{\sqrt{\varepsilon}}{\sqrt{\tau}} \int_{0,\cos^{-1}\tau}^{2\pi,2\pi-\cos^{-1}\tau} \frac{d\ell}{\sqrt{1 - (\cos \ell')/(\tau)}}$$

Substituting in Eq. 32

$$p_3 = \int_0^\infty g(E_o)\, E_o dE_o \int_{\cos \ell}^\infty \frac{d\tau\, f(\mu)}{\sqrt{\tau - \cos \ell}} \left(1 - \mu^2\right)^{5/2} \sqrt{\varepsilon}\, (1 + \varepsilon \cos \ell)^2 \qquad (34)$$

We wish now to evaluate Eq. 34 up to terms linear in ε, since higher order terms are not needed for our linearized equations.

We may solve Eq. 33 for $(1 - \mu^2)$

$$(1 - \mu^2) = \frac{1}{1 + \varepsilon\, I^2(\tau)}$$

We observe that $I^2(\tau) \longrightarrow \tau - \frac{a}{\tau} + \ldots$ as $\tau \longrightarrow \infty$ and goes to zero at $\tau = -1$ and is finite at all points between. Let us also expand

$$f(\mu) = \sum_{n=0}^{\infty} a_n \left(1 - \mu^2\right)^n$$

Then

$$p_3 = \sum_n a_n \int_0^\infty g(E_o)\, E_o dE_o \int_{\cos \ell}^\infty \frac{d\tau}{\sqrt{\tau - \cos \ell}} \sqrt{\varepsilon}\,(1 + \varepsilon \cos \ell)^2 \frac{1}{(1 + \varepsilon\tau)^{n+5/2}}$$

$$\cdot \frac{1}{\left[1 + \varepsilon\left(\tau - I^2(\tau)\right)/(1 + \varepsilon\tau)\right]^{n+5/2}} \qquad (35)$$

We could then expand the last term in Eq. 35, since

$$\frac{\varepsilon(\tau - I^2(\tau))}{1 + \varepsilon\tau} < \text{Const } \varepsilon$$

It is then clear that, aside from the 1, the other terms contribute to order $\varepsilon^{3/2}$, hence may be neglected.

We now reintroduce μ by

$$\varepsilon(\tau - \cos \ell) = (1 + \varepsilon \cos \ell) \frac{\mu^2}{1 - \mu^2} \tag{36}$$

and expanding in powers of ε, we get

$$p_3 = 2 \int_0^\infty g(E_o)\, E_o dE_o \left[\sum_n a_n \int_0^1 (1 - \mu^2)^{n+1} d\mu - \varepsilon \cos \ell \cdot \sum_n n a_n \int_0^1 (1 - \mu^2)^{n+1} d\mu \right] \tag{37}$$

The first term is clearly just p_{3_i}, the initial pressure. The perturbation term may be written, remembering $\delta b = B_i \varepsilon \cos \ell$

$$\delta p_3 = \frac{p_3}{2} \frac{\delta b}{B_i} \frac{\int_0^1 \left[(\partial f)/(\partial \mu)\right] \left[(1 - \mu^2)^2/(\mu)\right] d\mu}{\int_0^1 f(\mu)\,(1 - \mu^2)\, d\mu} = p_3 \frac{\delta b}{B_i} s \tag{38}$$

where s is a shape factor, dependent on the initial plasma angular distribution. s vanishes for an initially isotropic distribution and is positive for a distribution with motion predominantly along the field lines. Currently favored means of heating, by sidewise compression, may tend to make s negative.

We now return briefly to the derivation of the stability criterion in this case. Using Eqs. 6, 10, 38, and the fact that $\vec{\xi} \approx e^{i(kz \pm m\varphi)}$, we may write for the equations of equilibrium in the directions perpendicular to the field

$$\frac{\left[\nabla \times \nabla \times (\vec{\xi} \times \vec{B}_i)\right]}{4\pi} \times \vec{B}_i - s\,\frac{p_{3_i}}{B_i}\,\nabla \delta b$$

$$- k^2(p_{1_i} - p_{3_i})\,\frac{\left(\vec{B}_i \times \left[\vec{\xi} \times \vec{B}_i\right]\right)}{B_i^2} = 0 \qquad (39)$$

Here

$$\delta b = \frac{\vec{B}_i \cdot \nabla \times (\vec{\xi} \times \vec{B}_i)}{B_i}$$

As we have remarked before, equilibrium along the field lines is assured by a particle orbit method such as we are using.

We also note that ξ_z does not occur in our equations so we may choose it arbitrarily so that

$$\nabla \cdot \vec{\xi} = ik\,\xi_z + \nabla_2 \cdot \vec{\xi} = 0 \qquad (40)$$

where ∇_2 is the two-dimensional operator on r and φ.

We also note that

$$\nabla \times (\vec{\xi} \times \vec{B}_i) = (B_i \cdot \nabla)\,\vec{\xi} = B_i\,ik\,\vec{\xi}$$

The first term in Eq. 39 becomes on use of vector identities[2]

$$\frac{\left[\nabla \times \nabla \times (\xi \times B_i)\right] \times \vec{B}_i}{4\pi} = \frac{ikB_i}{4\pi}(\nabla \times \vec{\xi}) \times \vec{B}_i = \frac{ikB_i}{4\pi}\Big[(\vec{B}_i \cdot \nabla)\vec{\xi}$$

$$- \nabla(\vec{B}_i \cdot \vec{\xi})\Big] = -\frac{k^2 B_i^2}{4\pi}\vec{\xi} + \frac{B_i^2}{4\pi}\nabla\nabla_2 \cdot \vec{\xi}$$

Doing the same sort of manipulations on the rest of the equation and throwing away components in the z direction, we get

$$\left(\frac{B_i^2}{4\pi} + sp_3\right)\nabla\nabla_2 \cdot \vec{\xi} - k^2\vec{\xi}\left[\frac{B^2}{4\pi} + p_1 - p_3\right] = 0 \qquad (41)$$

The solution of Eq. 41 is

$$\vec{\xi}_2 = A\nabla\left[J_m(ik\gamma r)\, e^{i(kz \pm m\varphi)}\right] \qquad (42)$$

with $\gamma = \sqrt{\dfrac{B_i^2/4\pi + p_1 - p_3}{B_i^2/4\pi + sp_3}}$

Again the determination of A comes from Eq. 15, the condition that $\xi_r = \delta r_o e^{i(kz \pm m\varphi)}$.

$$A = \frac{\delta r_o}{ik\gamma J_m'(ik\gamma r_o)}$$

2. See, for example, Margenau and Murphy, Mathematics of Physics and Chemistry, p. 148.

Finally, we must determine

$$\delta P_i = \frac{B_i \,\delta b}{4\pi} + \delta p_3 = \delta b \left(\frac{B_i}{4\pi} + \frac{p_3 s}{B_i} \right)$$

$$\delta P_i = - \left(\frac{B_i^2}{4\pi} + p_3 s \right) \frac{k \,\delta r_o \gamma^2 \, J_m(ik\gamma r_o)}{i\gamma \, J_m'(ik\gamma r_o)} \; e^{i(kz \pm m\varphi)}$$

$$\delta P_i = - \left(\frac{B_i^2}{4\pi} + p_3 s \right) \delta \, (\gamma Y)^2 \, K_m(\gamma Y) \tag{43}$$

Using Eq. 18, we finally obtain the stability criterion

$$\left(1 + \frac{4\pi (p_1 - p_3)}{\alpha_p^2 \, B_o^2} \right) \alpha_p^2 \, Y^2 \, K_m(\gamma Y) + (m \pm \alpha_V Y)^2 \, \frac{G_{\beta,m}(Y) \; K_m(Y) - L_m(Y)}{1 - G_{\beta,m}(Y)} > 1$$

As we have remarked, this reduces to the magnetohydrodynamic result, Eq. 1, for an isotropic distribution. In practice, with heating by sidewise compression, the distribution is likely to be peaked towards $\mu = 0$. In general, this appears harmful to stability.

Of course, for α_p^2 close to unity, we may expect the magnetohydrodynamic theory to be valid.

The unfavorable results occurring from having p_3 large can be understood physically, since particles with μ close to zero are restricted by the mirror effect to the region of weak field and thus are reduced in energy by the perturbation.

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It is interesting to note that the "no-heat flow" approximation,[3]

$$\frac{d}{dt}\left(\frac{p_1 \; B^2}{\rho^3}\right) = \frac{d}{dt}\left(\frac{p_3}{\rho B}\right) = 0,$$

seems to give erroneous results in this case. It yields Eq. 2 with

$$\gamma = \sqrt{\frac{\left(\alpha_p^2 \; B_o^2/4\pi\right) + p_1 - p_3}{\left(\alpha_p^2 \; B_o^2/4\pi\right) + 2p_3 - \left(p_3^2/3p_1\right)}}\;,$$

a much more favorable result.

It is perhaps worth remarking explicitly that the calculations we have made refer to both ions and electrons. If they have the same initial angular distribution, then they will have the same density along the lines after the deformation, and no space charge will be developed. If their angular distribution is different, it would be necessary to modify the adiabatic invariants to include space charge forces. The effect of this is obscure to me.

There may also be some question whether surface effects have been treated adequately here. However, the change in energy of the surface must be small in the ratio of Larmor radius to radius of the pinch. The whole adiabatic invariant theory, of course, depends on small Larmor radius, so this is consistent with our assumptions.

We now turn to exploration of the meaning of Eq. 1.

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3. See footnote 1.

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VI. DISCUSSION OF RESULTS

First we consider briefly the nature of the functions appearing in Eq. 1.

$$K_m(Y) > 0; \qquad -L_m(Y) > 0$$

$$1 > G_{\beta,m}(Y) > 0; \qquad \frac{\partial G}{\partial \beta} = \frac{\partial G}{\partial Y} < 0 \tag{44}$$

Also, as may be verified by substitution, K and L both satisfy the differential equation

$$\frac{\partial f}{\partial Y} = \frac{1}{Y}\left[1 - (m^2 + Y^2)\, f^2\right] \tag{45}$$

As $Y \rightarrow \infty$ $K_m = -L_m \rightarrow \frac{1}{Y}$.

Also, as a rough indication of behavior, one may consider

$$K_m(Y) \approx -L_m(Y) \approx \frac{1}{\sqrt{m^2 + Y^2}} \tag{46}$$

This is quite good for $m \geq 2$, fair for $m = 1$ and poor for $m = 0$.

We now break up the discussion and consider cases of different m separately.

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A. $m = 0$

Here the equation becomes

$$\alpha_p^2 Y^2 K_o(Y) + (\alpha_V Y)^2 \frac{G_{\beta,o}(Y)\, K_o(Y) - L_o(Y)}{1 - G_{\beta,o}(Y)} > 1 \tag{47}$$

It is easy to show

$$\frac{d}{dY}\,(Y^2\, K_o) > 0.$$

We have also shown that the whole expression on the left of Eq. 47 has positive derivative at $Y = 0$ and $Y = \infty$ for all β. It seems plausible to assume this is so for all Y, though we have not proved it. Substituting $Y = 0$ in Eq. 47, we find, using the limiting forms of the Bessel functions,

$$\alpha_p^2 + \alpha_V^2/(\beta^2 - 1) > \frac{1}{2} \tag{48}$$

We may also, at this point, write down the trivial equation for the existence of an equilibrium

$$\alpha_p^2 + \alpha_V^2 < 1 \tag{49}$$

B. $m \geq 2$

Let us denote the left hand side of Eq. 1 as $F(\alpha_p, \alpha_V, m, Y, \beta)$. Then we note that

$$F(\alpha_p, \alpha_V, m, Y, \beta) \geq F(\alpha_p, \alpha_V, m, Y, \infty).$$

Hence, if the pinch is stable for $\beta = \infty$, it is certainly stable for all finite β. For $\beta = \infty$ the stability condition, Eq. 1, becomes

$$\alpha_p^2 Y^2 K_m(Y) - (m - \alpha_V Y)^2 L_m(Y) > 1 \qquad (50)$$

Since we are looking for the most unfavorable situation, it is sufficient to consider the -sign in $(m \pm \alpha_V Y)^2$.

At $Y = 0$, $L_m(Y) = -\frac{1}{m}$ so Eq. 50 is certainly satisfied. At $Y = \infty$, since $-L_m = K_m = 1/Y$, it is also satisfied. Hence, it is clear that for a given m and α_V, it is possible to find an α_{p_o} such that Eq. 50 is satisfied for all Y if $\alpha_p \geq \alpha_{p_o}$. Let us suppose we have done this for m = 2, i.e., we have found α_{p_o} such that

$$F_2, \alpha_V, \alpha_{p_o}(Y) > 1 \qquad (51)$$

for all Y.

We will now show that Eq. 50 must also be satisfied for all $m > 2$.

First we introduce $Y = \frac{mY'}{2}$ and rewrite Eq. 50 as

$$F_{m},\ \alpha_{V},\ \alpha_{p} = \alpha_p^2\, Y'^2 \left(\frac{m}{2}\right)^2 K_m\left(m\,\frac{Y'}{2}\right) - (2 - \alpha_V Y')^2 \left(\frac{m}{2}\right)^2 L_m\left(\frac{m}{2}\, Y'\right)$$

Hence, if we can show

$$\begin{aligned} &\left(\frac{m}{2}\right)^2 K_m\left(\frac{m}{2}\, Y'\right) > K_2(Y') \\ &- \left(\frac{m}{2}\right)^2 L_m\left(\frac{m}{2}\, Y'\right) > - L_2(Y') \end{aligned} \tag{52}$$

then Eq. 50 must be satisfied for all m, Y in virtue of the fact it is satisfied for m = 2.

Let $\left(\frac{m}{2}\right)^2 K_m\left(\frac{m}{2}\, Y\right) = k_m(Y)$.

From Eq. 45, we have

$$\frac{\partial k_m}{\partial Y} = \frac{1}{Y}\left[\left(\frac{m}{2}\right)^2 - (4 + Y^2)\, k_m^2\right]; \qquad \frac{\partial K_2}{\partial Y} = \frac{1}{Y}\left[1 - (4 + Y^2)\, K_2^2\right] \tag{53}$$

At Y = 0

$$K_2 = \frac{1}{2} \qquad k_m = \frac{m}{4}$$

Solving Eq. 53, we may write

$$k_m - K_2 = e^{-\int_0^Y (k+K)\, \frac{(4+Y^2)}{Y}\, dY} \left[\int_0^Y dY'' \left\{ \left(\frac{m}{2}\right)^2 - 1 \right\} \frac{1}{Y''} e^{\int_0^{Y''} (k+K)\, \frac{(4+Y'^2)}{Y'} dY'} + \frac{m}{4} - \frac{1}{2} \right] > 0 \tag{54}$$

A similar procedure may be applied to the consideration of L_m.

Thus we have proved Eq. 52 and hence need only satisfy Eq. 50 for $m = 2$ and $\beta = \infty$ to get a sufficient condition for stability of all higher m. The procedure used to do this is the same as that to be described for $m = 1$.

The results are shown in Table I.

Table I -- Sufficient Condition for $m \geq 2$ Stability

α_v	0	.1	.25	.5	1
α_p^2	.067	.122	.222	.462	1.260

By comparison with the figures of stability regions at the end of this report, it may be seen that any system stable against $m = 0, 1$ appears to be stable for the higher modes.

C. $m = 1$

For $m = 1$, our Eq. 1 becomes

$$\alpha_p^2 Y^2 K_1(Y^2) + (1 - \alpha_V Y)^2 \frac{G_{\beta,1}(Y)\, K_1(Y) - L_1(Y)}{1 - G_{\beta,1}(Y)} > 1 \qquad (55)$$

Again it may be easily seen by looking at the asymptotic forms that for any α_p, α_V, β, Eq. 55 must be satisfied at Y = 0 and ∞. At Y = 0, it is the effect of $G_{\beta,1}$, the external conductor, which makes for stability. At Y = ∞, it is the longitudinal fields. Due to the appearance of the factor $(1 - \alpha_V Y)^2$, there is a region of wavelength where the perturbation just fits the corkscrew of the external field tending to make for poor stability.

It is also easily seen from Eq. 55 that for a given α_V and β, one can find an α_{p_o} such that all greater values of α_p satisfy Eq. 55 for all Y. The method of calculation used was to fix α_V and β, then solve Eq. 55 for α_p^2 as a function of Y. The resulting curve gives $\alpha_p^2 = -\infty$ at Y = 0 and $\alpha_p^2 = -\alpha_V^2$ at Y = ∞, with a maximum in between. The value of this maximum is then α_{p_o}, since for this value of α_p, there is only one wavelength which is neutrally stable.

These calculations were performed by Mrs. Josephine Powers to whom many hearty thanks are due. Together with Eqs. 48 and 49, they define a region of stable pinch configuration. These regions are shown in Figs. 2-6, which show the α_p, β plane for α_V = 0, .1, .25, .5, and 1.

In conclusion, we are led to expect that quite reasonable compressions and fields lead to a stable configuration. In practice, of course, we must use a torus rather than an infinite cylinder to eliminate end losses, and one eventually must calculate in this geometry. From what we have seen, a magnetohydrodynamic calculation should suffice. It seems clear, however, that for tori with large major to minor radius, our results should apply.

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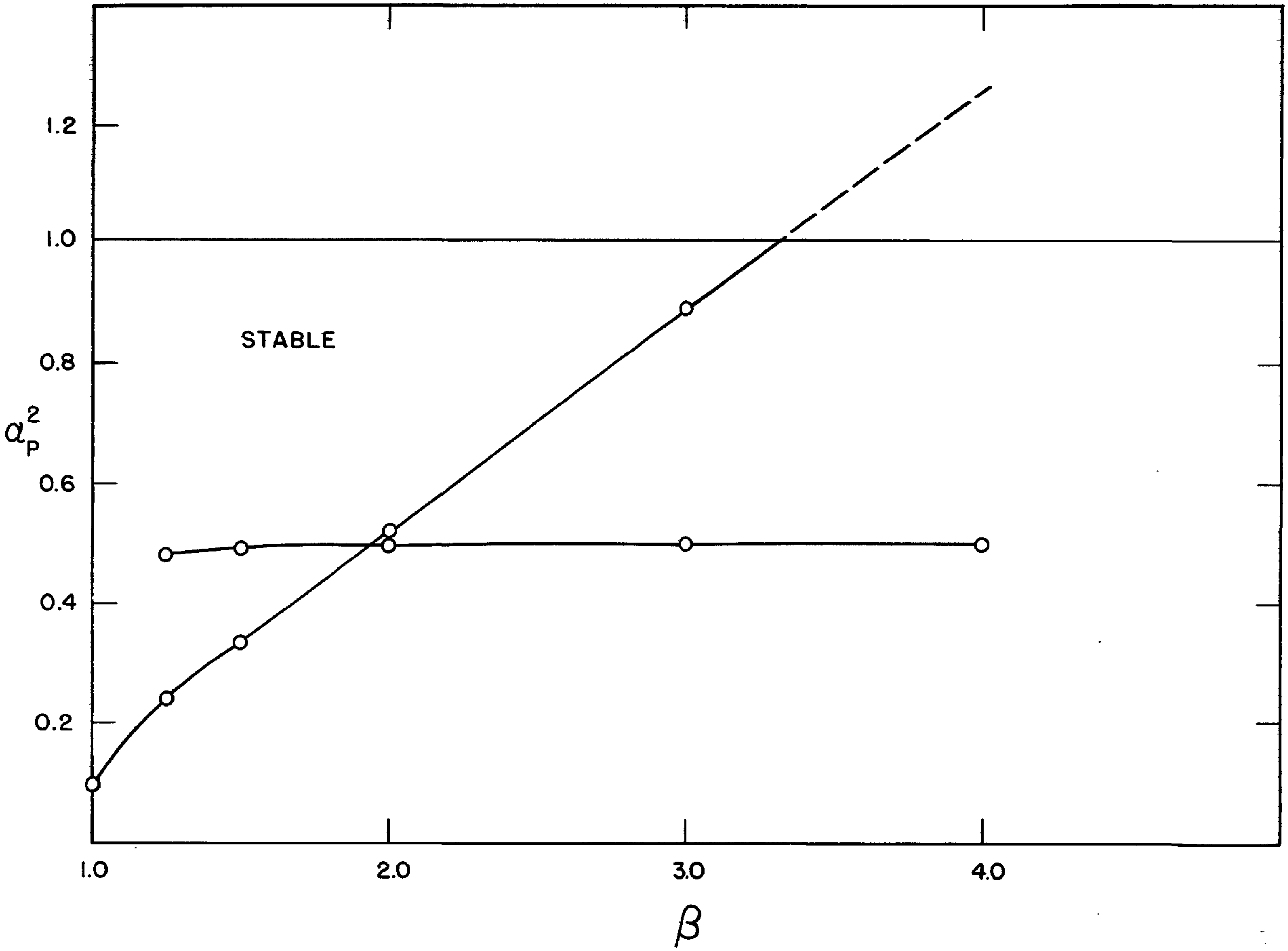

Fig. 2 Stability diagram; $\alpha_V = 0.1$

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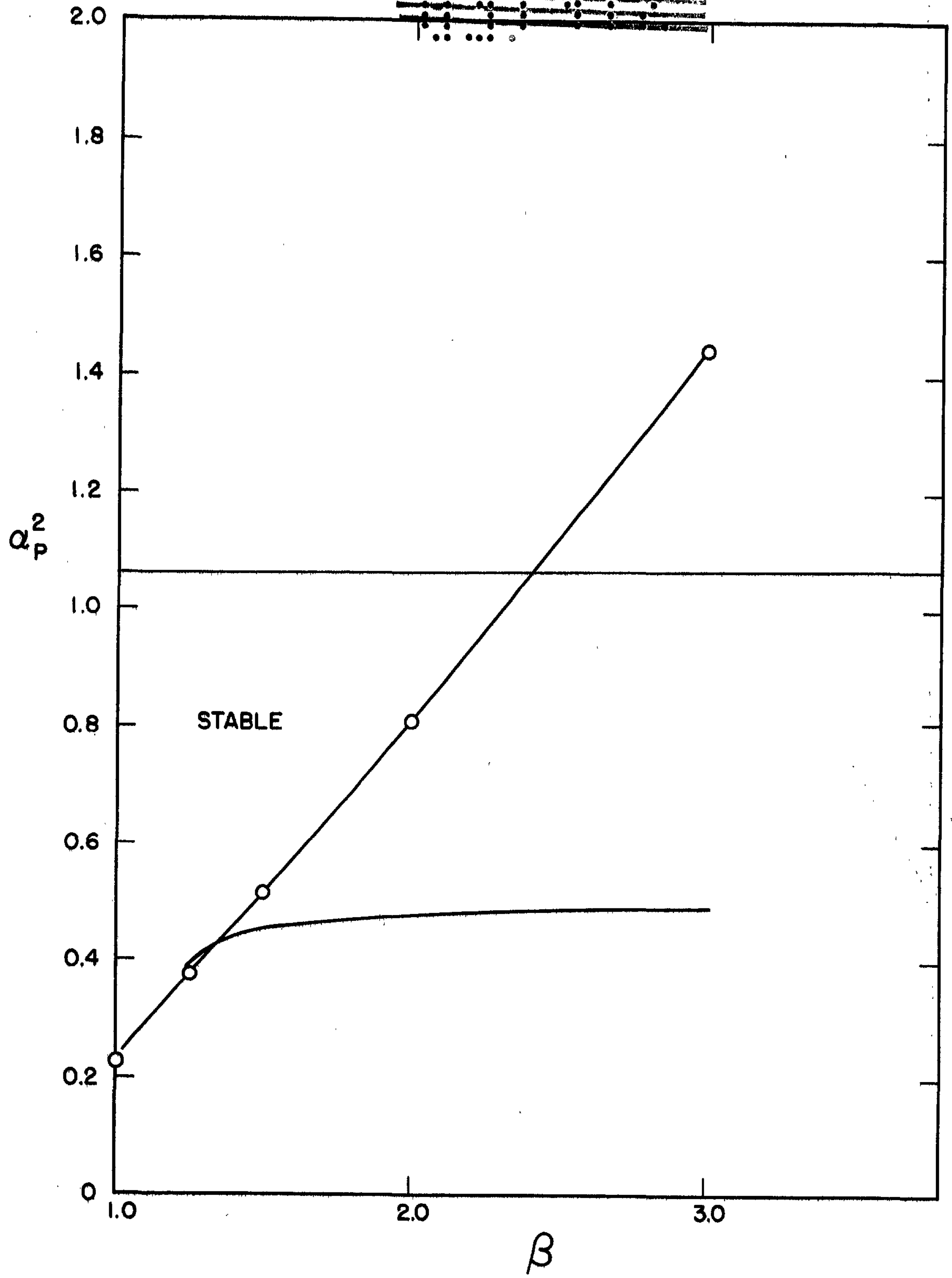

Fig. 3 Stability diagram; $\alpha_V = 0.25$

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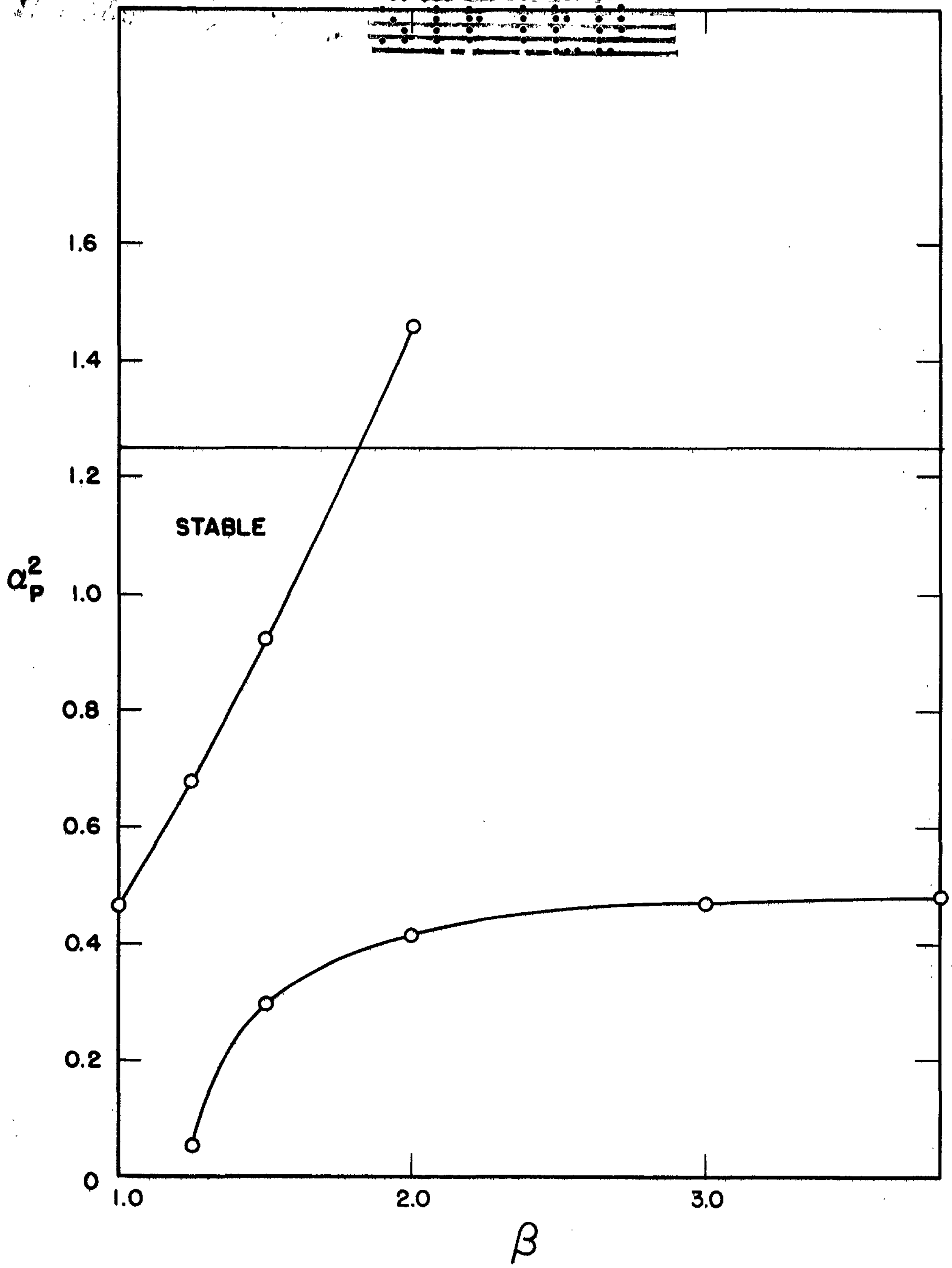

Fig. 4 Stability diagram; $\alpha_V = 0.5$

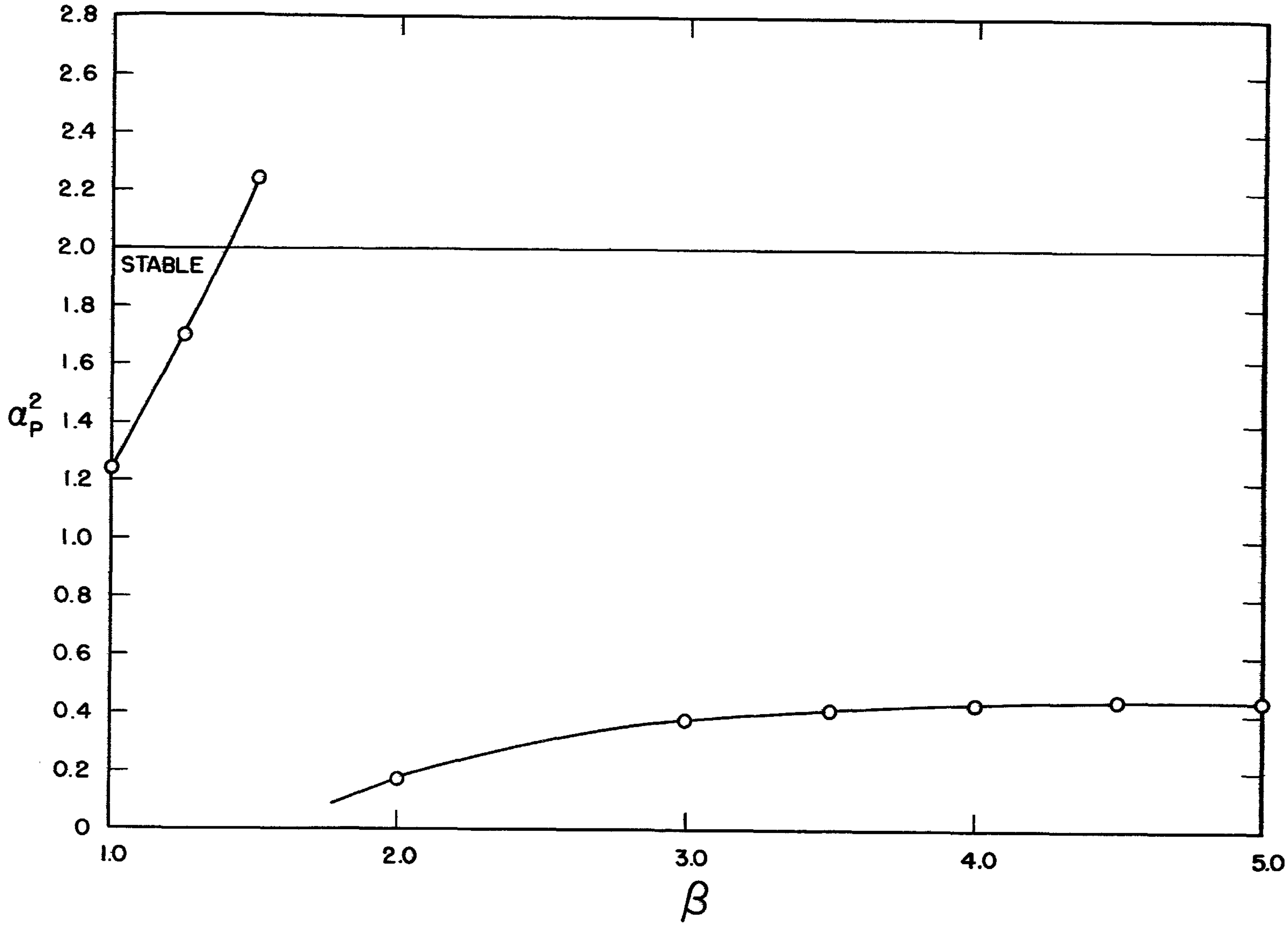

Fig. 5 Stability diagram; $\alpha_V = 1$

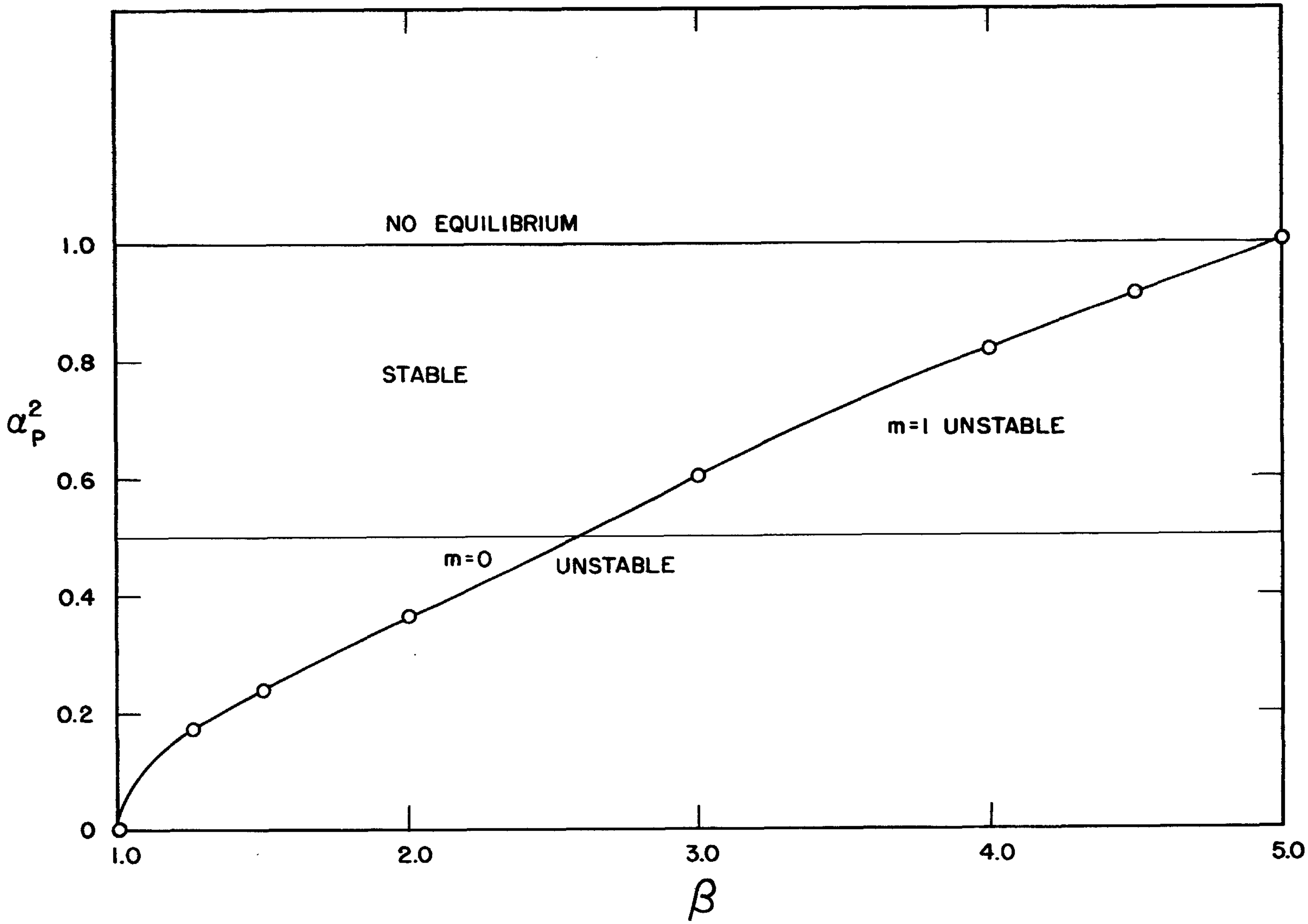

Fig. 6 Stability diagram; $\alpha_V = 0$

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